LATTICE CHARACTERIZATION OF CONVEX 3-POLYTOPES AND OF POLYGONIZATIONS OF 2-MANIFOLDS*

BY

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ABSTRACT

Let C be a polygonization of a 2-dimensional closed manifold without boundary, and $L(C)$ the set of all the faces of C, partially ordered by inclusion, with adjoinment of a minimal and a maximal element. Then $L(C)$ is a lattice, and its characterization is given here. Also a characterization of the lattice of the faces of a convex 3-polytope is given.

1. Introduction. The family $F(P)$ of all the faces of a d-dimensional polytope P, partially ordered by inclusion, is a lattice $L(P)$ of height $d+1$. The problem of characterizing these lattices, i.e., given a lattice L to decide whether L is (isomorphic to) the lattice $L(P)$ of some convex polytope P or not, is an open and difficult problem in the theory of convex polytopes. In this paper we solve the problem for the relatively simple case of 3-polytopes (Theorem 6). First we deal with the more general problem of characterizing the lattice of a polygonization of a 2-dimensional manifold. Then, using Steinitz's well-known theorem ([3], [5, chap. 13]), it is easy to characterize the lattice $L(P)$ of a 3-polytope P.

We use the terminology and notation of [4] and [5].

2. Definitions and notation.

DEFINITION 1. Let E be a topological space. A set $a \subset E$ is called *an i-cell in the wide sense*, briefly, an i-cell, if a is homeomorphic to an i-cell (in the usual sense).

A *2-dimensional topological cell complex* is a finite collection C of i-cells in the wide sense ($0 \leqq i \leqq 2$) in E such that:

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a) the intersection of any two cells in C is either empty or a cell in C;

b) for any cell a in C, the union of all the cells in C which are proper subsets of a, is the boundary of a (in the topological sense).

$|C|$ will denote the union of all the cells in C.

A particular case of a 2-dimensional topological cell complex is any collection C of cells (in the usual sense) in a Euclidean space E, which satisfies the above conditions. In this case we simply call C a 2-dimensional cell complex. An example is the 2-dimensional skeleton of any convex polytope.

DEFINITION 2. Let M be a closed, connected, 2-dimensional manifold without boundary (briefly: 2-manifold). A *polygonization* C of M is a 2-dimensional topological cell complex C such that $|C| = M$. In this case, each 2-cell (1-cell, 0-cell) of C is also called a face (edge, vertex) of C.

If M is also orientable, then it is known ([2, chap. III, section 7] and [6, pp. 141, 222, 322 note 36]) that M is embeddable in 3-dimensional Euclidean space R^3, and is characterized by a non-negative number, the genus of M. In this case, by "the genus of C" we mean the genus of M.

In the sequel, we use the theorem:

THEOREM 1. *Let C be a 2-dimensional topological cell complex. $|C|$ is a 2-manifold (closed, connected without boundary) if and only if*

a) *every 1-cell in C is contained in exactly two 2-cells of C;*

b) *for each 0-cell in C, all the 1-cells (a_i) and 2-cells (P_i) in C containing it may be cyclically ordered $a_1,\cdots,a_m$ and $P_1,\cdots,P_m$ in such a way that $a_i = P_i \cap P_{i-1}$ ($a_1 = P_1 \cap P_m$) and $m \geqq 3$.*

c) *C is not a union of two disjoint non-empty complexes.*

This statement can be found in [1, p. 46] for the case where all the 2-cells in C are simplices, and its generalization to our case is simple.

We shall now give another characterization of 2-dimensional topological cell-complexes which are manifolds (polygonizations), in which conditions b) and c) are replaced by a single lattice condition.

The family of all the faces of a d-dimensional polytope P, partially ordered by inclusion, is a lattice $L(P)$ of height $d+1$. The minimal element is the empty face, and the maximal element is the polytope P as a face of itself. In the same manner we define:

DEFINITION 3. The lattice $L(C)$ of a polygonization C is the set of all the

i-cells $(0 \leqq i \leqq 2)$ of C, partially ordered by inclusion, with adjoinment of a minimal element 0 and a maximal element I.

DEFINITION 4. A *proper* lattice is a lattice with a finite number of elements which satisfies the Jordan-Dedekind chain condition ([4, p. 11]), and in which every closed interval of height 2 is $\mathbf{2}^2$. ($\mathbf{2}^2$ is the lattice of the faces of a 1-polytope, i.e., of a segment. It is of height 2 and contains exactly 2 elements of height 1.)

If L is a proper lattice and $[\alpha, \beta]$ is a closed interval in L, we denote by $K_m^{\alpha\beta}$ the set of all the elements of L which are of height m and are contained in $[\alpha, \beta]$. The maximal (minimal) element of L is denoted I_L (O_L), and if there is no danger of confusion we write I (O). For $K_m^{0\alpha}$ and K_m^{0I} we write K_m^{α} and K_m, respectively.

The height of an element α of L is denoted $h(\alpha)$; the height of the lattice L is $h(L) = h(I)$. $\breve{L}$ is the lattice dual to L, and it is obvious that if L is proper, then so is $\breve{L}$. The (Euler) *characteristic* of a proper lattice L is

$$\chi(L) = \sum_{i=0}^{h(L)} (-1)^{i+1} \operatorname{card} K_i .$$

DEFINITION 5. Let L be a proper lattice, and γ an element of L such that $h(\gamma) \geqq 3$. We say that γ is *perforated* in L if there are disjoint and non-empty sets A, B such that $K_1^{\gamma} = A \cup B$, and $\alpha \vee \beta = \gamma$ for evety $\alpha \in A$, $\beta \in B$. We say that L is *unperforated* if in L and in $\breve{L}$ there are no perforated elements.

EXAMPLES. Let Δ_1, Δ_2 be disjoint triangles, and let L be the lattice of height 3 in which the elements of height 1 (2) are the vertices (edges) of Δ_1 and Δ_2, partially ordered by inclusion, with adjoinment of a minimal element 0 and a maximal element I. Then L is proper, I_L is perforated in L and $0_L (= I_{\breve{L}})$ is perforated in $\breve{L}$.

Let Δ_1, Δ_2 be 3-simplices whose intersection is a common vertex α. Let L be the lattice wbose elements are the i-dimensional faces $(0 \leqq i \leqq 2)$ of Δ_1 and of Δ_2, with adjoinment of a maximal and minimal element. Then L is proper, $h(L) = 4$ and α is perforated in L.

DEFINITION 6. Let L be a lattice. If there is a convex polytope the lattice of whose faces is (isomorphic to) L, we say that L is a P. L. (Polytope lattice).

THEOREM 2. *Let P be a d-polytope $(d \geqq 2)$ and L the lattice of (the faces of) P. Then L is proper and unperforated*, $h(L) = d + 1$, *and* $\chi(L) = 0$.

PROOF. It is obvious that $h(L) = d + 1$, the number of the elements of L is finite, and L satisfies the Jordan-Dedekind chain condition. By Euler's Theorem for polytopes ([5, page 130]) we have $\chi(L) = 0$. We prove that every closed interval of height 2 in L is $\mathbf{2}^2$ ·

If $\alpha \in L$, $\alpha < I$, we say that $[0,\alpha]$ is obtained from L by *truncation*. Obviously any lattice obtained from L by truncation is a P.L. (of the suitable face of P).

Let $[\alpha,\beta]$ be a closed interval of L. $[0,\beta]$ is a P.L., therefore $\overset{\frown}{[0,\beta]}$ is also a P.L. (of the polytope dual to β). $\overset{\frown}{[\alpha,\beta]}$ is obtained from $\overset{\frown}{[0,\beta]}$ by truncation and is therefore also a P.L. (of a polytope P'), and hence $[\alpha,\beta]$ is also a P.L. (of the polytope dual to P'). (See [5, p. 50, exercise 10].) In particular, if $h[\alpha,\beta] = 2$ then $[\alpha,\beta]$ is a P.L. of a 1-polytope, and is therefore 2^2.

The fact that L is unperforated follows directly from the fact that the 1-skeletons of P and the polytope dual to P are connected. Q. E. D.

In the sequel we shall see (Theorem 6) that for a lattice L of height 4 the converse of Theorem 2 is also true, thus we get a characterization of the lattices of (the faces of) 3-polytopes.

In the sequel we consider only proper lattices of height 4.

3. Polygonizations of 2-manifolds.

THEOREM 3. *If C is a 2-dimensional topological cell complex such that $|C|$ is a closed connected 2-manifold without boundary* (i.e. *C is a polygonization*), *then $L(C)$ is proper, of height* 4, *and unperforated.*

PROOF. $L(C)$ is proper because every edge in C contains exactly 2 vertices and because of condition a) in Theorem 1. Clearly $h(L(C)) = 4$.

The fact that $L(C)$ is unperforated follows easily from the fact that $|C|$ is connected and all the 2-cells in C which contain a given vertex can be cyclically ordered around that vertex (conditions b) in Theorem 1). Q. E. D.

The last theorem and the next one, in their combined form as Theorem 5, are the main result of this section. They are an equivalent form of Theorem 1 in lattice terminology, where conditions a, b, c of Theorem 1 are replaced by the condition that the lattice be proper and unperforated.

Theorem 4. *For any unperforated proper lattice L such that $h(L) = 4$ there exists a polygonization C such that $L(C)$ is isomorphic to L.*

PROOF. Let L be an unperforated proper lattice of height 4. We denote the elements of L of height 1, 2, 3 by α_i, β_i, γ_i respectively.

The fact that L is proper easily implies that card $K_1^\gamma \geqq 3$ for any $\gamma \in L$.

We now show that for any $\gamma \in L$ all the elements α_i in K_1^γ and all the elements β_i in K_2^γ may be cyclically ordered, $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_n$, so that $\alpha_i \vee \alpha_{i+1} = \beta_i$ for each $1 \leqq i \leqq n$ (addition of indices mod n).

Let $\gamma \in L$ and $\alpha_1 \in K_1^\gamma$. There is a β_1 such that $\alpha_1 < \beta_1 < \gamma$.

Since L is proper, $K_1^{\beta_1}$ contains exactly two atoms, one of them is α_1, the other we denote by $\alpha_2 \cdot K_2^{\alpha_2\gamma}$. contains exactly two elements; β_1 and β_2. $K_1^{\beta_2}$ contains two elements, one of them α_2 and the other, which is not α_1 (since $\alpha_1 \vee \alpha_2 = \beta_1 \neq \beta_2$), denoted α_3.

Assume that the sequences $\alpha_1, \cdots, \alpha_m$, $\beta_1, \cdots, \beta_{m-1}$ are already defined so that $\alpha_i \in K_1^\gamma$ for each $1 \leqq i \leqq m$, $\alpha_i \vee \alpha_{i+1} = \beta_i$ $(1 \leqq i < m)$ and $i \neq j \Rightarrow \alpha_i \neq \alpha_j$, $\beta_i \neq \beta_j$. We define α_{m+1} and β_m: $K_2^{\alpha_m\gamma}$ contains β_{m-1} and another element which we denote β_m. For each $1 \leqq i < m$ we have $\beta_i \neq \beta_m$, since if $\beta_m = \beta_k$ $(1 \leqq k < m)$ then $K_1^{\beta_m}$ contains α_k, α_{k+1}, α_m ($\beta_m \neq \beta_{m-1}$ implies $k+1 \neq m$ and therefore $\alpha_{k+1} \neq \alpha_m$) i.e., more than two elements, contradicting the fact that L is proper.

$K_1^{\beta_m}$ contains α_m and another element α_{m+1}. If $\alpha_k = \alpha_{m+1}$ for some $1 < k < m$, then $K_2^{\alpha_k\gamma}$ contains β_{k-1}, β_k, β_m, *i.e.*, more then two elements, contradicting the fact that L is proper. If $\alpha_{m+1} = \alpha_1$ then $m = \operatorname{card} K_1^\gamma$, i.e., the sequence $\alpha_1, \cdots, \alpha_m$ exhausts K_1^γ. For, if not, let

$$A = \{\alpha_i \mid 1 \leqq i \leqq m\}, \quad B = K^\gamma \setminus A.$$

A and B are disjoint and not empty. Since L is proper, it follows that for each $\alpha_i \in A$ and $\alpha \in B$ we have $h(\alpha_i \vee \alpha) > 2$, hence $\alpha_i \vee \alpha = \gamma$, i.e., γ is perforated in L, contradicting our assumption.

The result is a cyclic ordering $\alpha_1, \cdots, \alpha_n$ of all the elements of K_1^γ and a cyclic ordering $\beta_1, \cdots, \beta_n$ of elements of K_2^γ, such that $\alpha_i \vee \alpha_{i+1} = \beta_i$ for each i (addition of indices mod n).

We claim that the sequence $\beta_1, \cdots, \beta_n$ exhausts K_2^γ. If K_2^γ contains another element β, then $\alpha < \beta$ for some $\alpha \in K_1^\gamma$, and without loss of generality we assume $\alpha = \alpha_2$. But then $K_2^{\alpha_2\gamma}$ contains β, β_1, β_2, contradicting the fact that L is proper.

Therefore, if $\gamma_1, \cdots, \gamma_l$, are all the elements of height 3 in L, it is possible to associate with each γ_i a convex polygen P_i with card $K_1^{\gamma_i}$ edges.

Let M be the space consisting of all the P_i, with identification of common edges and vertices, and let C be the collection of all the cells in M which are images of the P_i's, their edges, and their vertices. Clearly, C is a 2-dimensional topological cell complex.

C satisfies condition a) of Theorem 1, since L is proper. Condition c) of Theorem 1 follows from the fact that I is unperforated in L. Condition b) of Theorem 1 follows easily from the fact that all the elements of height 3 in L are unperforated. The detailed proof is entirely analogous to the proof that with each $\gamma \in L$ it is possible to associate a convex polygon with card K_1^γ vertices.

Therefore M is a closed connected 2-dimensional manifold without boundary, hence C is a polygonization of M, and it is clear that $L(C)$ is isomorphic to L.

Q. E. D.

Let L be an unperforated proper lattice of height 4, and let $\gamma \in K_3$. From the proof of Theorem 4 it follows that one can define two orientations, which we call *opposite*, on the edges of K_2^γ:

$$\beta_1 \to \beta_2 \to \cdots \to \beta_n \to \beta_1 \text{ and } \beta_1 \leftarrow \beta_2 \leftarrow \cdots \leftarrow \beta_n \leftarrow \beta_1 .$$

DEFINITION 7. Let L be an unperforated proper lattice of height 4.

If it is possible, for each $\gamma \in K_3$, to define an orientation on the elements of K_2^γ in such a manner that the two orientations induced on each $\beta \in K_2$ by γ_1, γ_2 which satisfy $I > \gamma_2 > \beta < \gamma_1 < I$ are opposite, then we say that L is *orientable*. Otherwise — L is not orientable.

It is clear that if L is an orientable unperforated proper lattice of height 4 and C is a polygonization with $L(C)$ isomorphic to L, then $|C|$ is an orientable 2-manifold. Moreover, if $\chi(L) = -2g$ where g is a non-negative integer, then, by Euler's theorem for connected manifolds, the manifold $|C|$ is of genus g. The last theorems may then be summarized as follows:

THEOREM 5. *A lattice L is isomorphic to the lattice of a polygonization of a closed, connected, orientable 2-manifold without boundary of genus g if and only if L is proper, orientable, unperforated, of height 4 and $\chi(L) = -2g$.*

4. The lattice of a 3-polytope. The vertices and edges of a polygonization form a graph without double edges and without loops. In this graph we denote an edge with vertices a_1, a_2 by $(a_1 a_2)$, and a path through the vertices, $a_1, \cdots, a_n$ (in this order) by $(a_1, \cdots, a_n)$. (Here $(a_i a_{i+1})$ is an edge in the graph for each $1 \leqq i < n$). If $a_i \neq a_j$ for each $1 \leqq i, j < n$ $(i \neq j)$ and $a_n \neq a_i$ for each $1 < i < n$, the path is *simple*, otherwise it *cuts itself*. If $a_1 \neq a_n$, the vertices a_1, a_n are the *ends* of the path. A graph G is *connected* if for every two vertices a_1, a_2 in G $(a_1 \neq a_2)$ there is a path in G with ends a_1, a_2. G is *k-connected* if every subgraph of G obtained by deleting any $k-1$ vertices of G and the edges issuing of those vertices is connected.

Using this notation, we prove:

THEOREM 6. *A lattice L of height 4 is a P.L. if and only if L is proper, unperforated and $\chi(L) = 0$.*

PROOF. The statement "only if" follows directly from Theorem 2.

Let L be a proper, unperforated lattice of height 4 with $\chi(L) = 0$. There exist a 2-manifold M in R^3 and a polygonization C of M such that L is isomorphic to $L(C)$ (Theorem 5). The fact that M is in R^3 follows from the fact that $\chi(L) = 0$, for then M is orientable of genus 0, *i.e.*, M is a 2-sphere (see, e.g., [2, Chapter III, Section 7.2]).

Steinitz's theorem ([3], [5, chap. 13]) ensures that any planar and 3-connected graph is isomorphic to the graph of some 3-polytope. Let G be the graph of the vertices and edges of C. G is planar, so we need only to prove that G is also 3-connected.

Let a_1, a_2 be any two vertices in G, and G' the graph obtained from G be deleting a_1, a_2 and every edge which contains a_1 or a_2 or both. Let a_3, a_4 be any two different vertices in G'. We have to show that there exists a path in G' with the ends a_3 and a_4.

For each atom α_i in L let a_i denote the corresponding vertex in G. We claim that G contains a path with the ends a_3, a_4. In fact, let A be the set of all the atoms α_i in L such that G contains a path with ends a_i, a_3 ($\alpha_3 \in A$), and let $B = K_1 \setminus A$. (K_1 is the set of all the atoms in L.) Clearly, for each α_i, $\alpha_j \in A$ and for each $\alpha_k \in B$, a_i is connected to a_j by a path in G, and is not connected to a_k by a path in G. It follows from the proof of Theorem 4 that for each element γ of height 3 in L and for each $\alpha_i, \alpha_j \in K_1^\gamma$, a_i is connected to a_j by a path in G. Therefore, if B is not empty, then I is perforated in L —a contradiction.

Hence G contains a path l with ends a_3 and a_4, and we may assume that l is simple.

We distinguish five cases:

CASE a: $a_1, a_2 \notin l$.

Then l is a path in G' from a_3 to a_4.

CASE b: $a_1 \in l$, $a_2 \notin l$.

Assume $l = (a_3, \cdots, a_5, a_1, a_6, \cdots, a_4)$.

All the (2-dimensional) faces of the polygonization C which contain the vertex a_1 may be divided into two disjoint classes A and B separated by the path (a_5, a_1, a_6), so that for each edge $(a_1\ a_7)$ in C such that $a_6 \neq a_7 \neq a_5, (a_1\ a_7)$ belongs to two faces in the same class, and the edge $(a_1\ a_5)$ (and also $(a_1\ a_6)$) belongs to two faces which are not in the same class. Two faces c_1, c_2, one of which is in A and the other in B, have at most two vertices in common. For if γ_1, γ_2 are the elements of height 3 in L which correspond to c_1, c_2, then $\gamma_1 \wedge \gamma_2$ is an element of height at most 2 in L. Hence, since L is proper, $K_1^{\gamma_1 \wedge \gamma_2}$ contains at most 2 atoms. a_1 is one of the vertices common to c_1 and c_2; if they have another common element a_7, then necessarily $a_7 = a_5$ or $a_7 = a_6$. For if $a_6 \neq a_7 \neq a_5$, then $(a_1\ a_7)$ is not an edge in C, by the definition of the classes A, B, but then $\alpha_1 \vee \alpha_7 = \gamma_1$ and also $\alpha_1 \vee \alpha_7 = \gamma_2$, which is impossible in a lattice.

Therefore there is a class, say A, such that no face in A includes a_2. Now the path (a_5, a_1, a_6) may be replaced in an obvious way by a path from a_5 to a_6 through the vertices of the faces in A which does not include a_1. Let l' be the path obtained from l by this procedure. Then l' is a path in G' with ends a_3, a_4.

CASE c: $a_1 \notin l,\ a_2 \in l$.

As in case b, exchanging the roles of a_1, a_2.

CASE d: $a_1, a_2 \in l,\ a_1, a_2$ are not adjacent on l.

Assume $l = (a_3, \cdots, a_1, \cdots, a_7, a_2, a_8, \cdots, a_4)$. As in case b, the path (a_7, a_2, a_8) may be replaced by a path in G which is not through a_2. Thus l is reduced to a path of the type considered in case b.

CASE e: $a_1, a_2 \in l,\ a_1, a_2$ are adjacent on l.

Assume $l = (a_3, \cdots, a_1, a_2, a_8, \cdots, a_4)$. We replace the path (a_1, a_2, a_8) by a path $(a_1, \cdots, a_8)$ in G which is not through a_2, again reducing the argument to that of case b.

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